Reynolds stress under a change of frame of reference

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In this paper, we study the characteristics of the Reynolds stress under a change of frame, as defined by the Euclidean group of transformation. We show that being subject to the dynamical processes induced from the mean Navier-Stokes equations, the invariance group of the fluctuating velocity and the Reynolds stress is no longer the Euclidean group of transformation, which is merely a kinematical aspect, but reduces to the extended Galilean group of transformation. As a consequence, in contrast to developing the constitutive equations for the Cauchy stress in continuum mechanics, wherein the principle of material frame-indifference is a guiding principle, the frame-dependent kinematical quantities, e.g., the mean spin tensor, may be allowed to play an effective role as the constitutive variable in turbulence modeling.

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I. INTRODUCTION

In modern continuum mechanics, there are a few fundamental principles that apply to all bodies and motions. Among them, the principle of material frame-indifference is a guiding principle for constitutive equations, which express the relations between the stress and the motion of a body and thereby represent a variety of materials. The principle of material frame-indifference put forth by Noll [1] consists of two fundamental postulates. The first one states that the constitutive functional for the Cauchy stress in a continuous medium is form-invariant, i.e., it takes the same form, be it in an inertial frame or in a noninertial frame; and the second postulate asserts that the Cauchy stress is frame-indifferent, i.e., independent of the observers. In the community of turbulence research, the validity of this fundamental principle in continuum mechanics applied to turbulence modeling has long been an interesting but somewhat controversial topic (see Ref. [2]). Lumley [3] argued that the principle of material frame-indifference is not satisfied in turbulent flows, consequently it must be discarded. In fact, he considered a steady homogeneous pure plane strain in a steadily rotating framework and showed that the effect of rigid rotation on the Reynolds stress is serious.

In this paper, we shall study the characteristics of the fluctuating velocity under a change of frame from the perspectives of both kinematics and dynamics. We show that although kinematically the fluctuating velocity is frame-indifferent, being constrained by the dynamical processes stemming from taking the ensemble average on the Navier-Stokes equations, the invariance group of the fluctuating velocity is no longer the Euclidean group of transformation, but reduces to the extended Galilean group of transformation. Consequently, the Reynolds stress is not frame-indifferent, i.e., not invariant relative to the Euclidean group of transformation, but is merely invariant with respect to the extended Galilean group of transformation, a proper subgroup of the former. This gives not only a rigorous proof of the viewpoint of Lumley [3], namely that the Reynolds stress is not frame-indifferent, but it also demarks precisely the invariance group to which the Reynolds stresses belong. Furthermore, it is shown that two important quantities of turbulent flows—the turbulent kinetic energy K and the turbulent dissipation rate ϵ—are also invariant with respect to the extended Galilean group of transformation, but are not frame-indifferent in the sense of Noll [1].

II. CHARACTERISTICS OF THE REYNOLDS STRESS UNDER A CHANGE OF FRAME

In this paper, we consider an incompressible Navier-Stokes fluid with constant mass density ϱ and viscosity μ. The continuity equation and the linear momentum equation read

$$\operatorname{div}\mathbf{v}=0, \tag{1}$$

$$\varrho\dot{\mathbf{v}}=\operatorname{div}\mathbf{T}+\varrho\mathbf{B}, \tag{2}$$

where a dot denotes the material time derivative d/dt, $\mathbf{T}=-p\mathbf{1}+2\mu\mathbf{D}$, p is the pressure, $\mathbf{1}$ is the unit tensor, $2\mathbf{D}=\operatorname{grad}\mathbf{v}+(\operatorname{grad}\mathbf{v})^T$, and $\mathbf{B}$ is the body force density.

Taking an ensemble average on the above equations gives

$$\operatorname{div}\overline{\mathbf{v}}=0, \tag{3}$$

$$\varrho\frac{D\overline{\mathbf{v}}}{Dt}=\operatorname{div}(\overline{\mathbf{T}}+\tau)+\varrho\overline{\mathbf{B}}, \tag{4}$$

where an overbar represents the ensemble average, D/Dt denotes the material time derivative associated with the mean velocity field $\overline{\mathbf{v}}$, $\overline{\mathbf{T}}=-\overline{p}\mathbf{1}+2\mu\overline{\mathbf{D}}$, $2\overline{\mathbf{D}}=\operatorname{grad}\overline{\mathbf{v}}+(\operatorname{grad}\overline{\mathbf{v}})^T$, $\overline{\mathbf{B}}$ is the mean body force density, and $\tau:=-\varrho\overline{\mathbf{v}'\otimes\mathbf{v}'}$ is the Reynolds stress wherein $\mathbf{v}'$ is the fluctuating velocity, which gives rise to the so-called closure problem of turbulence modeling.

In the following, we shall study the kinematical and the dynamical properties of the Reynolds stress under a change of frame of reference, which means physically a change of observer.

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A. Kinematic characteristic of the Reynolds stress

Let us first consider a change of frame of reference (see Ref. [4]), from an inertial frame $\oint:(\mathbf{x},t)$ to a noninertial frame $\oint^*:(\mathbf{x}^*,t^*)$, which is defined by the Euclidean group of transformation,

$$\mathbf{x}^*(t^*)=\mathbf{Q}(t)\mathbf{x}+\mathbf{b}(t), \tag{5}$$

where $\mathbf{Q}(t)$ is an orthogonal tensor, $\mathbf{Q}(t)\mathbf{Q}(t)^T=\mathbf{Q}(t)^T\mathbf{Q}(t)=\mathbf{1}$, $\mathbf{b}(t)$ is an arbitrary vector of time, and $t^*=t+t_0$, where t_0 is a constant.

It is well known that the velocity field $\mathbf{v}(\mathbf{x},t)$ is not frame-indifferent, since under a change of frame

$$\mathbf{v}^*(\mathbf{x}^*,t^*)=\mathbf{Q}(t)\mathbf{v}(\mathbf{x},t)+\dot{\mathbf{Q}}(t)\mathbf{x}+\dot{\mathbf{b}}(t). \tag{6}$$

Taking an ensemble average on Eq. (6), we get

$$\overline{\mathbf{v}}^*(\mathbf{x}^*,t^*)=\mathbf{Q}(t)\overline{\mathbf{v}}(\mathbf{x},t)+\dot{\mathbf{Q}}(t)\mathbf{x}+\dot{\mathbf{b}}(t). \tag{7}$$

Subtracting Eq. (7) from Eq. (6) yields

$$\mathbf{v}^{*\prime}=\mathbf{Q}(t)\mathbf{v}'. \tag{8}$$

This shows that the fluctuating velocity $\mathbf{v}'$ is frame-indifferent. By Eq. (8), it is straightforward to show that the Reynolds stress $\tau:=-\varrho\overline{\mathbf{v}'\otimes\mathbf{v}'}$ is frame-indifferent as well, i.e.,

$$\tau^*=\mathbf{Q}(t)\tau\mathbf{Q}(t)^T. \tag{9}$$

This kinematical property of the Reynolds stress was shown by Speziale [5].

Remark 1. It is worthwhile to compare the fluctuating velocity $\mathbf{v}'$ and the Reynolds stress tensor $\tau:=-\varrho\overline{\mathbf{v}'\otimes\mathbf{v}'}$ in turbulence with the peculiar velocity $\mathbf{c}:=\mathbf{v}-\overline{\mathbf{v}}$ and the pressure tensor $\mathbf{P}:=\varrho\overline{\mathbf{c}\otimes\mathbf{c}}$ in the kinetic theory of gases (see Ref. [6]), and note that in fact they bear the same forms. In the kinetic theory of gases, $-\mathbf{P}$ is interpreted as the Cauchy stress tensor. However, there exists an obvious difference between the Cauchy stress and the Reynolds stress; for instance, on a solid boundary, the Reynolds stress is always zero due to the no-slip boundary condition of velocity, while even in a static state of flow the Cauchy stress is not so but simply reduces to a static pressure.

Now in view of Eqs. (8) and (9), it is clear that under a change of frame, the fluctuating velocity and the Reynolds stress are frame-indifferent, being a direct consequence of kinematics, and their invariance group is the Euclidean group of transformation, namely,

$$\mathbf{x}^*(t^*)=\mathbf{Q}(t)\mathbf{x}+\mathbf{b}(t). \tag{10}$$

Notwithstanding the kinematical properties as given in Eqs. (8) and (9), we shall show that, due to being subjected to the dynamical processes, the invariance group of the fluctuating velocity and thereby the Reynolds stress is not the Euclidean group of transformation (5)—a direct consequence of kinematics as seen—but is merely a proper subgroup of the Euclidean group of transformation.

B. Invariance group of the Reynolds stress subject to the dynamical process

It is generally accepted that turbulence is a continuum phenomenon, a feature of the flow itself rather than a material property of the fluid under consideration, such as its shear viscosity μ. Moreover, it is believed that the turbulent flows of a Newtonian fluid can be described by the Navier-Stokes equations, within the framework of continuum mechanics. In fact, for real turbulent flows, the evolution equation of the fluctuating velocity, which stems from the averaged Navier-Stokes equations, must be satisfied. This implies that in reality, not all fluctuating velocities $\mathbf{v}'$ which satisfy Eq. (8), a property directly derived from kinematics, i.e.,

$$\mathbf{v}^{*\prime}=\mathbf{Q}(t)\mathbf{v}', \tag{11}$$

are admissible in conformity with dynamics. The fluctuating velocity $\mathbf{v}'$ must satisfy its own evolution equation, i.e., it is subjected to the corresponding dynamical process that is induced adscititiously by taking the ensemble average on the Navier-Stokes equations. And this, as we shall see later, imposes a severe restriction on the invariance group of the fluctuating velocity under a change of frame and the invariance group of the Reynolds stress as well.

Now consider a change of frame from an inertial frame $\oint:(\mathbf{x},t)$ to a noninertial frame $\oint^*:(\mathbf{x}^*,t^*)$. We have from Eq. (2) that in $\oint$,

$$\varrho\mathbf{a}_{\oint}=\operatorname{div}\mathbf{T}+\varrho\mathbf{B},\ \ \mathbf{a}_{\oint}=\ddot{\mathbf{x}}; \tag{12}$$

and in $\oint^*$ it reads

$$\varrho^*\mathbf{a}_{\oint^*}=\operatorname{div}\mathbf{T}^*+\varrho^*\mathbf{B}^*, \tag{13}$$

where $\mathbf{T}^*=\mathbf{Q}(t)\mathbf{T}\mathbf{Q}(t)^T=-p^*\mathbf{1}^*+2\mu^*\mathbf{D}^*$, $\mu^*=\mu$, $\mathbf{B}^*=\mathbf{Q}(t)\mathbf{B}$, and

$$\mathbf{a}_{\oint^*}=\frac{d\mathbf{v}^*}{dt^*}-2\mathbf{A}(\mathbf{v}^*-\dot{\mathbf{b}})-\ddot{\mathbf{b}}(t)-(\dot{\mathbf{A}}-\mathbf{A}^2)[\mathbf{x}^*-\mathbf{b}(t)], \tag{14}$$

where $\mathbf{A}:=\dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T$.

Equation (14) can be rewritten as

$$\frac{d\mathbf{v}^*}{dt^*}=\frac{1}{\varrho^*}\operatorname{div}\mathbf{T}^*+\mathbf{B}^*+2\mathbf{A}(\mathbf{v}^*-\dot{\mathbf{b}})+\ddot{\mathbf{b}}(t)+(\dot{\mathbf{A}}-\mathbf{A}^2)[\mathbf{x}^*-\mathbf{b}(t)]. \tag{15}$$

Let $\mathbf{F}^*=\mathbf{B}^*+2\mathbf{A}(\mathbf{v}^*-\dot{\mathbf{b}})+\ddot{\mathbf{b}}(t)+(\dot{\mathbf{A}}-\mathbf{A}^2)[\mathbf{x}^*-\mathbf{b}(t)]$, which is called the apparent body force density (see Refs. [7,8]). Then Eq. (15) simply reads

$$\frac{d\mathbf{v}^*}{dt^*}=\frac{1}{\varrho^*}\operatorname{div}\mathbf{T}^*+\mathbf{F}^*, \tag{16}$$

where the apparent body force density $\mathbf{F}^*$ is not frame-indifferent. That is, under a change of frame

$$\mathbf{F}^* = \mathbf{B}^* + 2\mathbf{A}(\mathbf{v}^* - \dot{\mathbf{b}}) + \ddot{\mathbf{b}}(t) + (\dot{\mathbf{A}} - \mathbf{A}^2)[\mathbf{x}^* - \mathbf{b}(t)] \neq \mathbf{Q}(t)\mathbf{F}\mathbf{Q}(t)^T, \tag{17}$$

where $\mathbf{F} = \mathbf{B}$ in an inertial frame $\oint$.

In an inertial frame, Eq. (16) becomes

$$\frac{d\mathbf{v}}{dt} = \frac{1}{\varrho}\mathrm{div}\mathbf{T} + \mathbf{F}. \tag{18}$$

Now taking an ensemble average on Eq. (16), we have

$$\overline{\frac{d\mathbf{v}^*}{dt^*}} = \frac{1}{\varrho^*}\mathrm{div}\overline{\mathbf{T}^*} + \overline{\mathbf{F}^*}. \tag{19}$$

Subtraction of Eq. (19) from Eq. (16) gives the evolution equation of the fluctuating velocity $\mathbf{v}^{*'}$ in a noninertial frame $\oint^*$, noting that $\mathbf{F}^{*'} = \mathbf{B}^{*'} + 2\mathbf{A}\mathbf{v}^{*'}$,

$$\frac{D\mathbf{v}^{*'}}{Dt^*} + \overline{\mathbf{L}^*\mathbf{v}^{*'}} = \frac{1}{\varrho^*}\mathrm{div}\mathbf{T}^{*'} + \mathbf{F}^{*'} + \mathbf{G}^*$$

$$= \frac{1}{\varrho^*}\mathrm{div}\,\mathbf{T}^{*'} + \mathbf{B}^{*'} + 2\mathbf{A}\mathbf{v}^{*'} + \mathbf{G}^*, \tag{20}$$

where $\overline{\mathbf{L}^*} = \mathrm{grad}\,\overline{\mathbf{v}^*}$, $\mathbf{T}^{*'} = -p^{*'}\mathbf{1}^* + 2\mu^*\mathbf{D}^{*'}$, $2\mathbf{D}^{*'} = \mathrm{grad}\mathbf{v}^{*'} + (\mathrm{grad}\mathbf{v}^{*'})^T$, and $\mathbf{G}^* = \mathrm{div}(\mathbf{v}^{*'}\otimes\mathbf{v}^{*'} - \overline{\mathbf{v}^{*'}\otimes\mathbf{v}^{*'}})$.

Note that in an inertial frame the apparent body force density fluctuation $\mathbf{F}^{*'}$ becomes $\mathbf{F}' = \mathbf{B}'$. The evolution equation of the fluctuating velocity $\mathbf{v}'$ in $\oint$ reads

$$\frac{D\mathbf{v}'}{Dt} + \overline{\mathbf{L}}\mathbf{v}' = \frac{1}{\varrho}\mathrm{div}\mathbf{T}' + \mathbf{F}' + \mathbf{G}$$

$$= \frac{1}{\varrho}\mathrm{div}\mathbf{T}' + \mathbf{B}' + \mathbf{G}, \tag{21}$$

where $\overline{\mathbf{L}} = \mathrm{grad}\overline{\mathbf{v}}$, $\mathbf{T}' = -p'\mathbf{1} + 2\mu\mathbf{D}'$, $2\mathbf{D}' = \mathrm{grad}\mathbf{v}' + (\mathrm{grad}\mathbf{v}')^T$, and $\mathbf{G} = \mathrm{div}(\mathbf{v}'\otimes\mathbf{v}' - \overline{\mathbf{v}'\otimes\mathbf{v}'})$.

Comparing Eqs. (20) and (21), we know immediately that the evolution equation of the fluctuating velocity $\mathbf{v}'$ is *not frame-indifferent*, because of the occurrence of $\mathbf{A} := \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T = -\mathbf{A}^T$, which is the spin of the noninertial frame $\oint^*$ with respect to the inertial frame $\oint$. This spin, as seen in Eq. (20), affects the evolution of the fluctuating velocity $\mathbf{v}'$.

Let Σ be defined as

$$\Sigma := \frac{D\mathbf{v}^{*'}}{Dt^*} + \overline{\mathbf{L}^*}\mathbf{v}^{*'} - \frac{1}{\varrho^*}\mathrm{div}\,\mathbf{T}^{*'} - \mathbf{F}^{*'} - \mathbf{G}^* = \mathbf{0}. \tag{22}$$

Then $\overline{\Sigma\otimes\mathbf{v}^{*'} + \mathbf{v}^{*'}\otimes\Sigma}$ results in the Reynolds stress transport equation in a noninertial frame $\oint^*$,

$$\frac{D\tau^*}{Dt^*} + \overline{\mathbf{L}^*}\tau^* + \tau^*\overline{\mathbf{L}^*}^T = \frac{1}{\varrho^*}\{\overline{(\mathrm{div}\mathbf{T}^{*'})\otimes\mathbf{v}^{*'}} + \overline{\mathbf{v}^{*'}\otimes\mathrm{div}\mathbf{T}^{*'}}\} - \mathrm{div}\overline{\mathbf{v}^{*'}\otimes\mathbf{v}^{*'}\otimes\mathbf{v}^{*'}} + \overline{\mathbf{B}^{*'}\otimes\mathbf{v}^{*'}} + \overline{\mathbf{v}^{*'}\otimes\mathbf{B}^{*'}} + 2\mathbf{A}\tau^* + 2\tau^*\mathbf{A}^T. \tag{23}$$

And in an inertial frame $\oint$, Eq. (23) becomes

$$\frac{D\tau}{Dt} + \overline{\mathbf{L}}\tau + \tau\overline{\mathbf{L}}^T = \frac{1}{\varrho}\{\overline{(\mathrm{div}\mathbf{T}')\otimes\mathbf{v}'} + \overline{\mathbf{v}'\otimes\mathrm{div}\mathbf{T}'}\} - \mathrm{div}\overline{\mathbf{v}'\otimes\mathbf{v}'\otimes\mathbf{v}'} + \overline{\mathbf{B}'\otimes\mathbf{v}'} + \overline{\mathbf{v}'\otimes\mathbf{B}'}. \tag{24}$$

From Eqs. (23) and (24), it is clear that like the evolution equation of the fluctuating velocity $\mathbf{v}^{*'}$, the Reynolds stress transport equation is *not frame-indifferent* either, due to the same reason, namely the occurrence of the spin $\mathbf{A} = \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T$ under a change of frame. The same is true for any higher-order moment equations constructed from the evolution equation of the fluctuating velocity $\mathbf{v}^{*'}$.

Now let us study the invariance group of the fluctuating velocity $\mathbf{v}'$ under a change of frame, being subjected to the dynamical processes. From Eqs. (20) and (21), we know that under a change of frame from $\oint:(\mathbf{x},t)$ to $\oint^*:(\mathbf{x}^*,t^*)$, the evolution equation of $\mathbf{v}'$,

$$\frac{D\mathbf{v}'}{Dt} + \overline{\mathbf{L}}\mathbf{v}' = \frac{1}{\varrho}\mathrm{div}\mathbf{T}' + \mathbf{B}' + \mathbf{G}, \tag{25}$$

becomes

$$\frac{D\mathbf{v}^{*'}}{Dt^*} + \overline{\mathbf{L}^*\mathbf{v}^{*'}} - 2\mathbf{A}\mathbf{v}^{*'} = \frac{1}{\varrho^*}\mathrm{div}\mathbf{T}^{*'} + \mathbf{B}^{*'} + \mathbf{G}^*. \tag{26}$$

Also, from Eq. (8), we know that under a change of frame, the kinematical property of the fluctuating velocity $\mathbf{v}'$ reads

$$\mathbf{v}^{*'} = \mathbf{Q}(t)\mathbf{v}'. \tag{27}$$

Now apply $\mathbf{Q}(t)$ to Eq. (25) on both sides, making use of Eq. (27) and noting that $\mathbf{T}^{*'} = \mathbf{Q}(t)\mathbf{T}'$, $\varrho^* = \varrho$, and $\mathbf{B}^{*'} = \mathbf{Q}(t)\mathbf{B}'$. It follows that

$$\mathbf{Q}(t)\left(\frac{D\mathbf{v}'}{Dt} + \overline{\mathbf{L}}\mathbf{v}'\right) = \frac{1}{\varrho^*}\mathrm{div}\mathbf{T}^{*'} + \mathbf{B}^{*'} + \mathbf{G}^*. \tag{28}$$

A comparison of Eq. (28) with Eq. (26) then yields

$$\frac{D\mathbf{v}^{*'}}{Dt^*} + \overline{\mathbf{L}^*}\mathbf{v}^{*'} - 2\mathbf{A}\mathbf{v}^{*'} = \mathbf{Q}(t)\left(\frac{D\mathbf{v}'}{Dt} + \overline{\mathbf{L}}\mathbf{v}'\right). \tag{29}$$

This shows that the evolution equation of $\mathbf{v}'$ is *invariant* if and only if $\mathbf{A} = \dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T = \mathbf{0}$, that is, $\mathbf{Q}(t) = \mathbf{Q}_0$, where $\mathbf{Q}_0$ is an arbitrary constant orthogonal tensor. In other words, we have shown that, being subject to the dynamical processes, the invariance group of the fluctuating velocity $\mathbf{v}'$ is not the Euclidean group of transformation as defined by Eq. (10), but reduces to its proper subgroup as follows:

$$\mathbf{x}^\star = \mathbf{Q}_0\mathbf{x} + \mathbf{b}(t), \tag{30}$$

which may be called the extended Galilean group of transformation, noting that when $\mathbf{b}(t) = \mathbf{V}_0 t$, where $\mathbf{V}_0$ is a constant velocity, the Galilean group of transformation is obtained. Therefore, we have the following.

Theorem. Being subject to the evolution Eq. (25), a dynamical process induced by taking the ensemble average on the Navier-Stokes equations, the invariance group of the fluctuating velocity $\mathbf{v}'$ is the extended Galilean group of transformation, a proper subgroup of the Euclidean group of transformation.

Consequently, it is a simple matter to show that the Reynolds stress is also extended Galilean invariant, but not frame-indifferent (Euclidean invariant) in the sense of Truesdell and Noll [4]. That is, we have

$$\tau^\star = \mathbf{Q}_0 \tau \mathbf{Q}_0^T \tag{31}$$

under the extended Galilean group of transformation (30). Hence, we have the following.

Corollary 1. The Reynolds stress τ is not frame-indifferent under a change of frame, as defined by the Euclidean group of transformation, but invariant with respect to the extended Galilean group of transformation.

In addition, we have the following result concerning two important quantities in the theory of turbulence, the turbulent kinetic energy and the turbulent dissipation rate.

Corollary 2. The turbulent kinetic energy $K := \frac{1}{2}\varrho\overline{\mathbf{v}'\cdot\mathbf{v}'}$ and the turbulent dissipation rate $\epsilon := 2\mu\overline{\mathbf{D}'\cdot\mathbf{D}'}$, where $\mathbf{D}' = \frac{1}{2}[\mathrm{grad}\mathbf{v}' + (\mathrm{grad}\mathbf{v}')^T]$ are not frame-indifferent but invariant with respect to the extended Galilean group of transformation.

Proof. The proof of K is obvious by taking the trace of the Reynolds stress tensor. From the preceding theorem, we know grad$\mathbf{v}'$ is not frame-indifferent but invariant with respect to the extended Galilean group of transformation. There follows readily the proof of ϵ based on its definition. Q.E.D.

Since by the theorem the Reynolds stress is not frame-indifferent under a change of frame, but invariant relative to the extended Galilean group of transformation, it follows immediately that in contrast to developing the constitutitve equations for the Cauchy stress in continuum mechanics, the frame-dependent kinematical quantities, e.g., the mean spin tensor $\overline{\mathbf{W}} = \frac{1}{2}[\mathrm{grad}\overline{\mathbf{v}} - (\mathrm{grad}\overline{\mathbf{v}})^T]$, which is not frame-indifferent, may be allowed to play a role as the constitutive variable in turbulence modeling.

Remark 2. In the constitutive theory of continuum mechanics, the spin tensor $\mathbf{W} = \frac{1}{2}[\mathrm{grad}\mathbf{v} - (\mathrm{grad}\mathbf{v})^T]$ is excluded from being a constitutive argument of the constitutive functional for the Cauchy stress, since it is not frame-indifferent. Under a change of frame, as defined by the Euclidean group of transformation,

$$\mathbf{W}^* = \mathbf{Q}(t)\mathbf{W} + \mathbf{A}. \tag{32}$$

However, the spin tensor $\mathbf{W}$ is invariant under the extended Galilean group of transformation, $\mathbf{x}^\star = \mathbf{Q}_0\mathbf{x} + \mathbf{b}(t)$, namely

$$\mathbf{W}^\star = \mathbf{Q}_0\mathbf{W}. \tag{33}$$

And of course, so is the mean spin tensor $\overline{\mathbf{W}}$. Therefore, the above theorem and its corollaries provide the rationale for the justification of the constitutive assumption that the mean spin tensor $\overline{\mathbf{W}}$ should be included as a constitutive argument in developing closure models for turbulence.

Now let us review briefly some constitutive equations for fluids developed in history. First, we mention a nonlinear constitutive equation for fluid propsed by Stokes [9], which for the Cauchy stress $\mathbf{T}$ takes the form

$$\mathbf{T} = -p\mathbf{1} + \mathcal{F}(\mathbf{D}), \tag{34}$$

where $\mathbf{D}$ is the stretching tensor and where $\mathcal{F}(\mathbf{0}) = \mathbf{0}$. Assuming the fluid to be isotropic, for the case when $\mathcal{F}$ is linear, he worked out a linear constitutive equation as follows:

$$\mathbf{T} = -p\mathbf{1} + \lambda(\mathrm{tr}\,\mathbf{D})\mathbf{1} + 2\mu\mathbf{D}. \tag{35}$$

This is now called the Navier-Stokes fluid (see Ref. [10]), and the resulting dynamical equations are called the Navier-Stokes equations, which are assumed to hold true even when the fluid is in turbulence.

In an attempt to generalize the hypothesis given by Stokes, Boussinesq [11] replaced Eq. (34) by an apparently more general constitutive equation, which reads

$$\mathbf{T} = -p\mathbf{1} + \mathcal{F}(\mathbf{D}, \mathbf{W}), \tag{36}$$

where $\mathbf{W}$ is the spin tensor and where $\mathcal{F}(\mathbf{0}, \mathbf{W}) = \mathbf{0}$.

However, Noll [12] showed that the principle of material frame-indifference reduces Boussinesq's [11] constitutive equation (36) to Eq. (34) given by Stokes [9], since the frame-dependent spin tensor $\mathbf{W}$ has to be dropped out as a constitutive argument, and the constitutive functional $\mathcal{F}$ must be isotropic. Therefore, in view of the constitutive theory of continuum mechanics, in fact, no generalization at all was made by Boussinesq [11].

Yet, as we have remarked, in the constitutive theory of turbulence closure modeling, since the mean spin tensor $\overline{\mathbf{W}}$ is invariant relative to the extended Galilean group of transformation, it is allowed to serve as a constitutive argument in the constitutive equations for the Reynolds stress. Therefore, in contrast to the case of continuum mechanics, the constitutive equation proposed by Boussinesq [11] indeed would become a more general model for the Reynolds stress than Stokes' model [9], if the Cauchy stress $\mathbf{T}$ were replaced by the Reynolds stress τ in both Eqs. (34) and (36), if p were replaced by $\frac{2}{3}K$, and if the stretching tensor $\mathbf{D}$ and the spin tensor $\mathbf{W}$ were replaced by their means correspondingly. Namely, it follows that the model for the Reynolds stress corresponding to Stokes' model reads

$$\tau = -\tfrac{2}{3}K\mathbf{1} + \mathcal{F}(\overline{\mathbf{D}}), \tag{37}$$

where $\mathcal{F}(\mathbf{0})=\mathbf{0}$, while the model for the Reynolds stress corresponding to Boussinesq's model reads

$$\tau=-\frac{2}{3}K\mathbf{1}+\mathcal{F}(\overline{\mathbf{D}},\overline{\mathbf{W}}), \tag{38}$$

where $\mathcal{F}(\mathbf{0},\mathbf{0})=\mathbf{0}$.

Finally, it is worth noting the difference between modeling the Cauchy stress in continuous media and modeling the Reynolds stress in turbulence. In continuum mechanics, the Cauchy stress is assumed to be frame-indifferent, whereas the Reynolds stress is not frame-indifferent, as we have shown. It is invariant only relative to the extended Galilean group of transformation, as seen in Eq. (31). As a consequence, the frame-dependent kinematical quantities such as **W** are excluded from being the constitutive variables for the Cauchy stress, but are allowed to play an effective role in modeling the Reynolds stress. This significant difference, as it stands, may be regarded as an expression of the fact that the Cauchy stress in fact represents the intrinsic properties of the materials (see Ref. [4]), which are independent of the observers; however, by nature, the Reynolds stress, despite its name, depicts merely the complex phenomena of the turbulent flows, which may depend on the observers from an inertial frame of reference to a noninertial one.

III. CONCLUDING REMARKS

We have shown in this paper that being subjected to the dynamical processes induced by taking the ensemble average on the Navier-Stokes equations, the invariance group of the velocity fluctuation, the Reynolds stress, the turbulent kinetic energy, and the turbulent dissipation rate is the extended Galilean group of transformation, but not the Euclidean group of transformation, which is simply a kinematical property. In other words, these quantities are frame-dependent, not frame-indifferent in the sense of Truesdell and Noll [4], but invariant relative to the Galilean group of transformation.

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